Static Auction Design: Private Values

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Allocations and Sales Everywhere

- Sale of house, car, airline tickets.
- Sale of IPL teams/players.
- Sale of spectrum bandwidth, coal mining rights.
- Allocation of ancestral property.
- Allocation of a public good - park, museum, library.

Allocation Problems

- One or more objects are transferred from the seller to the buyers.
- Monetary transfers are involved.

Outline and Objectives

- Focus on environment with transfers where agents evaluate transfers using quasilinear utility functions.
- Mechanism design using single object auctions.
- Focus on structural properties of possible mechanisms.
- Bilateral trading.

Single Object Allocation

- A single object needs to be allocated to a finite set of agents N.
- Transfers/Payments are allowed.
- Agents utility is *quasilinear*: value for the object minus payment (transfer amount)
  - Transfers can be potentially positive, negative, or zero.

Private Information

- An agent's value for the object is his **private** information.
- Value of agent i is known to agent i completely but not known to other agents or to the seller/planner.
- The value/type of agent i is denoted by v_i.
- If agent i with type v_i gets the object with probability α_i and pays p_i, then his net utility is

$$\alpha_i v_i - p_i.$$

Two Decisions

Two decisions:

- Allocation decision: who gets the object with what probability,
- payment decision: transfer amount of each agent.

Allocation and payment decisions depend on the objectives of the designer.

Information Aggregation

Asks each agent to *report* his type and based on that makes allocation and payment decisions.

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Restricting attention to such direct mechanisms is without loss of generality.

The Domain

- Assumption: The type of each agent i is a non-negative real number in $V_i \equiv [0, \beta_i]$. The seller has zero value for the object.
- The seller (designer) knows V_i for each i - hence, no report can be made outside V_i.
- A type profile is a collection of types of all the agents $v \equiv (v_1, \ldots, v_n)$.
- The set of all type profiles or the domain of the problem is: $V \equiv V_1 \times \ldots \times V_n$.

The Direct Mechanism

A **direct mechanism** is a collection of pairs $\{a_i, p_i\}_{i \in N}$, where

- **allocation rule.** $a_i : V \to [0, 1]$ is the allocation probability of agent i satisfying $\sum_{j \in N} a_j(v) \leq 1$ for all $v \in V$
- **payment rule.** $p_i : V \to \mathbb{R}$ is the payment amount of agent i.

Incentives

Can we design the allocation rules and payment rules such that each agent has the incentive to report his **true** type to the direct mechanism?

Dominant Strategy Incentive Compatibility

Definition

A mechanism $\{a_i, p_i\}_{i \in N}$ *is* **dominant strategy incentive compatible (DSIC)** *if for every agent* $i \in N$*, for every* $v_{-i} \in V_{-i}$*, and for every* $v_i, v_i' \in V_i$*, we have*

$$a_i(v_i, v_{-i})v_i - p_i(v_i, v_{-i}) \geq a_i(v_i', v_{-i})v_i - p_i(v_i', v_{-i}).$$

First-Price Auction is Not DSIC

- Highest reported type gets the object and pays his type. Others pay zero.
- Suppose $N = \{1, 2\}$. If agent 2 reports 8 and agent 1 has value 10, he has no incentive to report more than 8.
- Truthtelling is not a **dominant strategy**.

Vickrey (Second-price) Mechanism

- Highest reported type gets the object with probability one (ties broken in some way).
- Agents who do not get the object pay zero and the winner of the object pays the second highest reported value.

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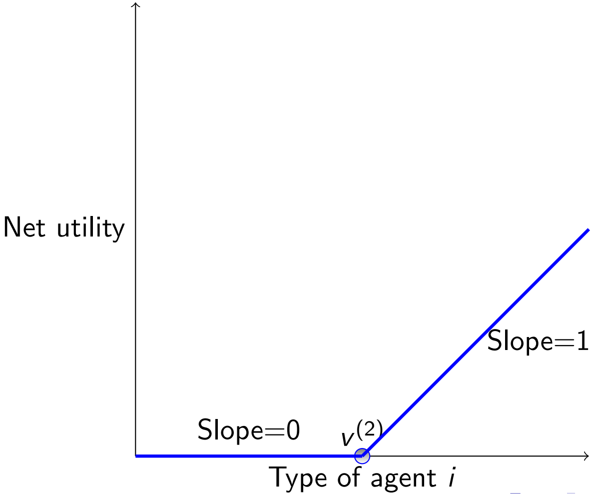
Theorem

The Vickrey mechanism is DSIC.

Closer Look at Net Utility

Fix agent i and his true type v_i. Fix others report at v_{-i}.

Let $\mathcal{U}^{Vick}(v_i, v_{-i})$ be the net utility of agent i from truthtelling in the Vickrey auction at (v_i, v_{-i}).



Two Observations

- Utility function is non-decreasing and convex.
- Derivative (wherever exists) is equal to the allocation probability.

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- Derivative (wherever exists) is equal to the allocation probability.

All DSIC mechanisms have this property.

Question

Can we characterize the set of all DSIC mechanisms?

Facts from Convex Analysis - Rockafellar's book

Let $g : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval.

Definition

A function $g : I \to \mathbb{R}$ *is* **convex** *if for every* $x, y \in I$ *and for every* $\lambda \in (0, 1)$*, we have*

$$\lambda g(x) + (1 - \lambda) g(y) \geq g(\lambda x + (1 - \lambda) y).$$

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- A convex function is continuous in the interior of its domain.
- Further, a convex function is differentiable *almost everywhere* in its domain.
  - More formally, there is a subset of $I' \subseteq I$ such that I' is dense in I, $I \setminus I'$ has measure zero and g is differentiable at every point in I'.

Subgradient

Definition

For any $x \in I$, x^* *is a* **subgradient** *of a convex function* $g : I \to \mathbb{R}$ *at* x *if*

$$g(z) \geq g(x) + x^*(z - x) \qquad \forall\, z \in I.$$

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If g is differentiable at $x \in I$, denote the derivative of g at x as $g'(x)$.

Lemma

Suppose $g : I \to \mathbb{R}$ is a convex function. Suppose x is in the interior of I and g is differentiable at x, then $g'(x)$ is the unique subgradient of g at x.

More Facts on Subgradients

Lemma

Suppose $g : I \to \mathbb{R}$ is a convex function. Then for every $x \in I$, the subgradient of g at x exists.

The set of subgradients of g at a point $z \in I$ is denoted as $\partial g(z)$.

Non-decreasing Subgradient

Lemma

Suppose $g : I \to \mathbb{R}$ is a convex function. Let $\phi : I \to \mathbb{R}$ such that $\phi(z) \in \partial g(z)$ for all $z \in I$. Then, for all $x, y \in I$ such that $x > y$, we have $\phi(x) \geq \phi(y)$.

By definition,

$$g(x) \geq g(y) + \phi(y)(x - y)$$
$$g(y) \geq g(x) + \phi(x)(y - x).$$

Adding these two inequalities, we get

$$(x - y)(\phi(x) - \phi(y)) \geq 0.$$

Since $x > y$, we get $\phi(x) \geq \phi(y)$.

Illustration

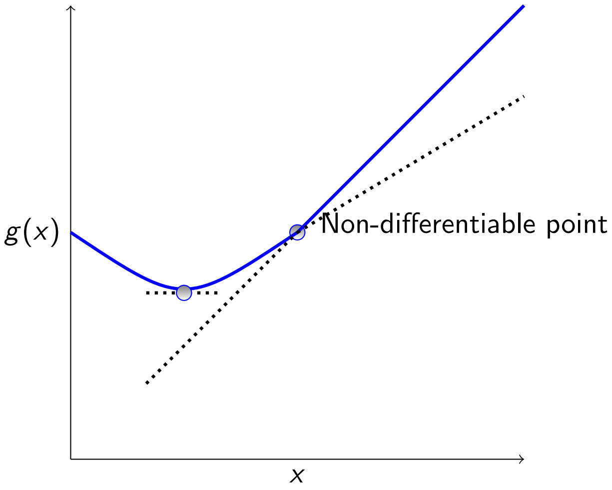


Figure: A convex function and its subgradients

Fundamental Theorem of Convex Analysis

Lemma

Let $g : I \to \mathbb{R}$ be a convex function. Then, for any $x, y \in I$,

$$g(x) = g(y) + \int_y^x \phi(z) dz,$$

where $\phi : I \to \mathbb{R}$ is a map satisfying $\phi(z) \in \partial g(z)$ for all $z \in I$.

Back to DSIC Mechanisms

Definition

A mechanism $M \equiv \{a_i, p_i\}_{i \in N}$ *is* **dominant strategy incentive compatible (DSIC)** *if for every agent* $i \in N$*, for every* $v_{-i} \in V_{-i}$*, and for every* $v_i, v_i' \in V_i$*, we have*

$$a_i(v_i, v_{-i})v_i - p_i(v_i, v_{-i}) \geq a_i(v_i', v_{-i})v_i - p_i(v_i', v_{-i}).$$

Rephrasing DSIC

Definition

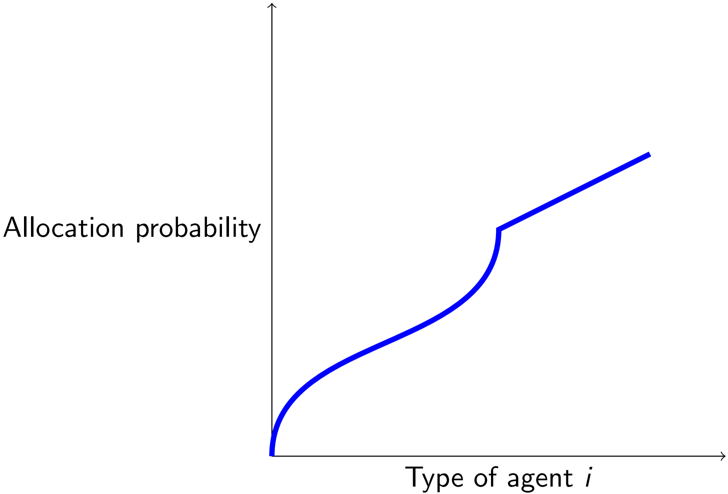
A mechanism $M \equiv \{a_i, p_i\}_{i \in N}$ *is* **dominant strategy incentive compatible (DSIC)** *if for every agent* $i \in N$*, for every* $v_{-i} \in V_{-i}$*, and for every* $v_i, v_i' \in V_i$*, we have*

$$\mathcal{U}^M(v_i, v_{-i}) \geq \mathcal{U}^M(v_i', v_{-i}) + a_i(v_i', v_{-i})\big[v_i - v_i'\big].$$

Monotone Allocation Rules

Definition

An allocation rule a_i is **non-decreasing** *if for every $v_{-i} \in V_{-i}$ we have $a_i(v_i, v_{-i}) \geq a_i(v_i', v_{-i})$ for all $v_i, v_i' \in V_i$ with $v_i' < v_i$.*



Main Characterization - Myerson

Theorem

A mechanism $M \equiv \{a_i, p_i\}_{i \in N}$ *is DSIC if and only if*

- **Monotone.** a_i *is non-decreasing for all* $i \in N$
- **Revenue Equivalence.** *for all* $i \in N$, *for all* $v_{-i} \in V_{-i}$, *and for all* $v_i \in V_i$

$$\mathcal{U}_i^M(v_i, v_{-i}) = \mathcal{U}_i^M(0, v_{-i}) + \int_0^{v_i} a_i(x_i, v_{-i}) dx_i.$$

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$$p_i(v_i, v_{-i}) = p_i(0, v_{-i}) + v_i a_i(v_i, v_{-i}) - \int_0^{v_i} f_i(x_i, v_{-i}) dx_i.$$

Sketch of Proof

Easy direction:

$$\mathcal{U}_i^M(v_i', v_{-i}) - \mathcal{U}_i^M(v_i, v_{-i}) = \int_{v_i}^{v_i'} a_i(x_i, v_{-i}) dx_i \leq (v_i' - v_i) a_i(v_i', v_{-i}).$$

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Difficult direction:

- DSIC M implies $\mathcal{U}_i^M(\cdot, v_{-i})$ is convex for each v_{-i}.
- $a_i(v_i, v_{-i})$ is a subgradient of this convex function at v_i. So, it is non-decreasing.
- Then, fundamental theorem of convex analysis gives revenue equivalence.

Main Implications for Design

- A DSIC mechanism must involve a non-decreasing allocation rule.
- The payment at a type is uniquely determined by the payment at the lowest type and the allocation rule.

Revenue Equivalence Re-examined

Suppose $\{a_i, p_i\}_{i \in N}$ and $\{a_i, p_i'\}_{i \in N}$ are two DSIC mechanisms. Revenue equivalence says for every $i \in N$ and for every v_{-i},

$$p_i(v_i, v_{-i}) - p_i'(v_i, v_{-i}) = p_i(0, v_{-i}) - p_i'(0, v_{-i}).$$

for all v_i.

Revenue Equivalence Re-examined

Suppose $\{a_i, p_i\}_{i \in N}$ and $\{a_i, p'_i\}_{i \in N}$ are two DSIC mechanisms. Revenue equivalence says for every $i \in N$ and for every v_{-i},

$$p_i(v_i, v_{-i}) - p'_i(v_i, v_{-i}) = p_i(0, v_{-i}) - p'_i(0, v_{-i}).$$

for all v_i.

A Different DSIC Mechanism:

- Object goes to the highest reported type.
- Winner pays zero, but losers receive a transfer equal to the winner's reported type.
- This mechanism is DSIC, allocation rule same as Vickrey mechanism. Payments must differ by a constant amount from the Vickrey mechanism.

The Implementation Perspective

Designer's objectives are encoded in *allocation rules* $\{a_i\}_{i \in N}$.
Payment rules are means to achieve incentives.

Definition

A collection of allocation rules $\{a_i\}_{i \in N}$ *are* **implementable** *(in dominant strategies) if there exists payments rules* $\{p_i\}_{i \in N}$ *such that* $\{a_i, p_i\}_{i \in N}$ *is a DSIC mechanism.*

What Allocation Rules are Implementable?

Theorem

A collection of allocation rules $\{a_i\}_{i \in N}$ *is implementable if and only if each* a_i *is non-decreasing.*

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- Highest reported type gets the object.
- Highest reported type among those types which are above a *reserve price* wins the object - if there are no such types, object is unsold.
- Ex-ante weights are assigned to agents, and highest weighted type wins the object.
- Many more . . ..

Sketch of Proof

- Suppose $\{a_i\}_{i \in N}$ is implementable. Then there is some $\{p_i\}_{i \in N}$ such that $\{a_i, p_i\}_{i \in N}$ which is DSIC. Earlier result says that each a_i is non-decreasing.
- Suppose each a_i is non-decreasing. Then, fixing $p_i(0, v_{-i})$ at some value for each i and each v_{-i}, and setting $p_i(v_i, v_{-i})$ for each (v_i, v_{-i}) as before, defines a DSIC mechanism.

Deterministic Mechanisms and Allocation Rules

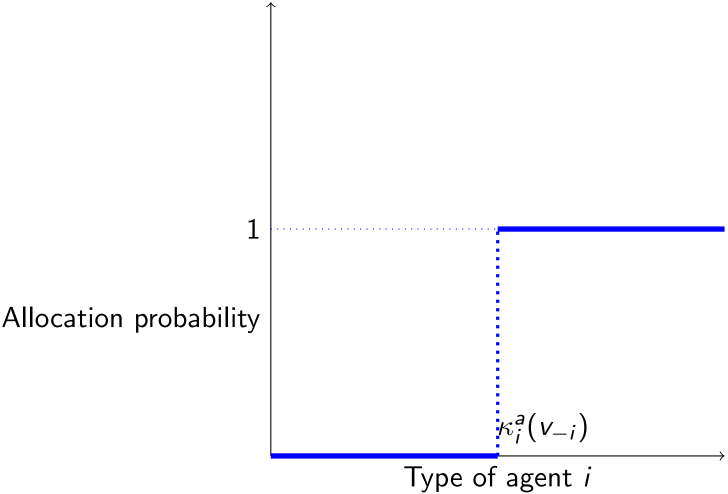

Figure: A deterministic implementable allocation rule

Cutoff Payments

$$\kappa_i^a(v_{-i}) = \begin{cases} \inf\{v_i \in V_i : a_i(v_i, v_{-i}) = 1\} & \text{if } a_i(v_i, v_{-i}) = 1 \text{ for some } v_i \\ 0 & \text{otherwise} \end{cases}$$

- The infimum amount one needs to report to start winning the object.
- Every non-decreasing allocation rule can be implemented by setting

$$p_i(v_i, v_{-i}) = \kappa_i^a(v_{-i})$$

if $a_i(v_i, v_{-i}) = 1$ and

$$p_i(v_i, v_{-i}) = 0$$

if $a_i(v_i, v_{-i}) = 0$.

Optimal Auction Design

What auction maximizes expected revenue of the seller?

Prior Information

- Each agent $i \in N$ draws his value from $V_i = [0, \beta_i]$ using a probability distribution with cdf G_i (density g_i). All draws are independent and G_{-i} is cummulative distribution of all the agents except agent i.
- Consider a mechanism $M \equiv \{a_i, p_i\}_{i \in N}$. An agent i with type v_i has an **interim allocation probability** of

$$\alpha_i(v_i) := \int_{V_{-i}} a_i(v_i, v_{-i}) dG_{-i}(v_{-i}).$$

and an **interim payment** of

$$\pi_i(v_i) := \int_{V_{-i}} p_i(v_i, v_{-i}) dG_{-i}(v_{-i}).$$

Weakening DSIC

Requiring DSIC is too demanding - truthtelling is best irrespective of what other agents do.

What if truthtelling is only a Bayesian Nash equilibrium - if others report truthfully, your expected utility is maximized by telling the truth.

Bayesian incentive compatibility - every agent must maximize expected utility from truthtelling.

Bayesian Incentive Compatibility (BIC)

Definition

A mechanism $M \equiv \{a_i, p_i\}_{i \in N}$ *is* **Bayesian incentive compatible (BIC)** *if for every agent* $i \in N$ *and for every* $v_i, v_i' \in V_i$*, we have*

$$\alpha_i(v_i)v_i - \pi_i(v_i) \geq \alpha_i(v_i')v_i - \pi_i(v_i').$$

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Expected utility from truthtelling $\mathbb{U}_i^M(v_i) := \alpha_i v_i - \pi_i(v_i)$.

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$$\mathbb{U}_i^M(v_i) \geq \mathbb{U}_i^M(v_i') + \alpha_i(v_i')(v_i - v_i').$$

Myerson - BIC Version

Definition

An allocation rule a_i is called **non-decreasing in expectation (NDE)** *if* $\alpha_i(v_i) \geq \alpha_i(v_i')$ *for all* $v_i, v_i' \in V_i$ *with* $v_i' < v_i$.

Myerson - BIC Version

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Theorem

A mechanism $M \equiv \{a_i, p_i\}_{i \in N}$ is BIC if and only if

- **Monotone.** *a_i is NDE for all $i \in N$*
- **Revenue Equivalence.** *for all $i \in N$ and for all $v_i \in V_i$*

$$\pi_i(v_i) = \pi_i(0) + v_i\alpha_i(v_i) - \int_0^{v_i} \alpha_i(x_i)dx_i.$$

$$OR$$

$$\mathbb{U}_i^M(v_i) = \mathbb{U}_i^M(0) + \int_0^{v_i} \alpha_i(x_i)dx_i.$$

Bayesian Implementation

Theorem

A collection of allocation rules $\{a_i\}_{i \in N}$ *is implementation in Bayes-Nash equilibrium if and only if it is NDE.*

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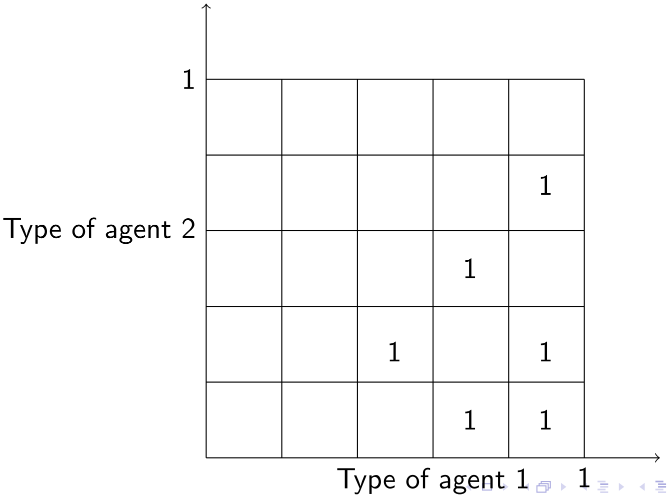

BIC-DSIC Equivalence

Theorem (Manelli-Vincent, 2007)

Suppose M is a BIC mechanism. Then, there exists a DSIC mechanism M' such that for all $i \in N$ and for all $v_i \in V_i$,

$$\mathbb{U}_i^{M}(v_i) = \mathbb{U}_i^{M'}(v_i).$$

Bayesian Revenue Equivalence

Two payment rules implementing the same allocation rule in Bayes-Nash equilibrium must differ by a constant in expectation.

Bayesian Revenue Equivalence

Two payment rules implementing the same allocation rule in Bayes-Nash equilibrium must differ by a constant in expectation.

First-price and second-price auctions use the same allocation rule with the same expected payment at the lowest type. Hence, they must generate the same expected payment for every type of agent.

Individual Rationality

Definition

A mechanism M is **interim individually rational (IIR)** *if for every $i \in N$ and for every $v_i \in V_i$, we have*

$$\mathbb{U}_i^M(v_i) \geq 0.$$

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$$\mathbb{U}_i^M(v_i) \geq 0.$$

Usually applied for BIC mechanisms. Since for BIC mechanisms, interim expected utilities are non-decreasing, IIR is equivalent to requiring

$$\mathbb{U}_i^M(0) \geq 0.$$

Expected Revenue from a Mechanism

Expected payment of agent i from mechanism $M \equiv \{a_i, p_i\}_{i \in N}$ is

$$\int_0^{\beta_i} \pi_i(v_i) g_i(v_i) dv_i.$$

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Expected revenue from mechanism $M \equiv \{a_i, p_i\}_{i \in N}$ is

$$R^M := \sum_{i \in N} \int_0^{\beta_i} \pi_i(v_i) g_i(v_i) dv_i.$$

Optimal Mechanism

A mechanism $M \equiv \{a_i, p_i\}_{i \in N}$ is **optimal** if it is BIC, IIR, and for any other BIC and IIR mechanism M', we have

$$R^M \geq R^{M'}.$$

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A mechanism $M \equiv \{a_i, p_i\}_{i \in N}$ is **optimal** if it is BIC, IIR, and for any other BIC and IIR mechanism M', we have

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Very sensitive to distributional assumption.

Payment of Agent i

$$\int_0^{\beta_i} \pi_i(v_i)g_i(v_i)dv_i = \pi_i(0) + \int_0^{\beta_i} \alpha_i(v_i)v_i g_i(v_i)dv_i - \int_0^{\beta_i}\int_0^{v_i} \big(\alpha_i(s_i)ds_i\big) g_i(s_i)ds_i,$$

Payment of Agent i

$$\int_0^{\beta_i} \pi_i(v_i) g_i(v_i) dv_i = \pi_i(0) + \int_0^{\beta_i} \alpha_i(v_i) v_i g_i(v_i) dv_i - \int_0^{\beta_i} \int_0^{v_i} \big(\alpha_i(s_i) ds_i\big) g_i(s_i) ds_i,$$

Changing the order of integration in the last term

$$\int_0^{\beta_i} \int_0^{v_i} \big(\alpha_i(s_i) ds_i\big) g_i(v_i) dv_i = \int_0^{\beta_i} \Big(\int_{v_i}^{\beta_i} g_i(s_i) ds_i \Big) \alpha_i(v_i) dv_i = \int_0^{\beta_i} (1 - G_i(v_i)) \alpha_i(v_i) dv_i.$$

Rewriting Revenue

$$R^M := \sum_{i \in N} \pi_i(0) + \sum_{i \in N} \int_0^{\beta_i} \left(v_i - \frac{1 - G_i(v_i)}{g_i(v_i)}\right) \alpha_i(v_i) g_i(v_i) dv_i.$$

We now define the **virtual valuation** of agent $i \in N$ with valuation $v_i \in V_i$ as

$$w_i(v_i) = v_i - \frac{1 - G_i(v_i)}{g_i(v_i)}.$$

Simplifying

$$
\begin{aligned}
R^M &= \sum_{i \in N} \pi_i(0) + \sum_{i \in N} \int_0^{\beta_i} w_i(v_i)\alpha_i(v_i) g_i(v_i) dv_i \\
&= \sum_{i \in N} \pi_i(0) + \sum_{i \in N} \int_0^{\beta_i} \left(\int_{V_{-i}} a_i(v_i, v_{-i}) g_{-i}(v_{-i}) dv_{-i} \right) w_i(v_i) g_i(v_i) dv_i \\
&= \sum_{i \in N} \pi_i(0) + \sum_{i \in N} \int_V w_i(v_i) a_i(v) g(v) dv \\
&= \sum_{i \in N} \pi_i(0) + \int_V \left[\sum_{i \in N} w_i(v_i) a_i(v) \right] g(v) dv.
\end{aligned}
$$

Implication of IIR

$\mathbb{U}_i^M(0) \geq 0$ or $\pi_i(0) \leq 0$. Maximizing revenue implies that optimal mechanism must have $\pi_i(0) = 0$ for all i.

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$$\max_{a_1,\ldots,a_n} \int_V \left[\sum_{i \in N} w_i(v_i) a_i(v_i, v_{-i}) \right] g(v) dv$$
$$\text{subject to } \; a_i \text{ is NDE for each } i.$$

Expected Virtual Value Maximization

Theorem

The allocation rule in an optimal mechanism maximizes the total expected virtual valuation among all Bayes-Nash implementable allocation rules.

Solving the Relaxed Problem

- Solve the optimization problem (expected virtual value maximization) without NDE constraint.
- Can be done by point-wise maximization - choose a rule that picks the highest virtual value agent as long as the virtual value is non-negative.
- At every type profile $v \equiv (v_1, \ldots, v_n)$, we assign
  - $a_i(v) = 0$ for all $i \in N$ if $w_i(v_i) < 0$ for all $i \in N$;
  - else $a_i(v) = 1$ for some $i \in N$ such that $w_i(v_i) \geq w_j(v_j)$ for all $j \neq i$.

Regular Distributions

Definition

A virtual valuation w_i of agent i is **regular** *if for all $s_i, v_i \in T_i$ with $s_i > v_i$, we have $w_i(s_i) > w_i(v_i)$.*

- Standard distributions are regular.
- A sufficient condition for regularity is that the *hazard rate* of the distribution is regular.
- If the distribution is regular, then the optimal solution of the relaxed problem becomes optimal in the constrained problem.
- Without regularity, a procedure called *ironing* is needed to derive optimal mechanism.

Optimal Mechanism

Theorem

Suppose the regularity holds for each agent. Consider the following allocation rule $\{a_i^*\}_{i \in N}$*. For every type profile* $v \equiv (v_1, \dots, v_n) \in V$,

$$a_i^*(v) = 0 \text{ if } w_i(v_i) < 0 \; \forall \; i \in N,$$

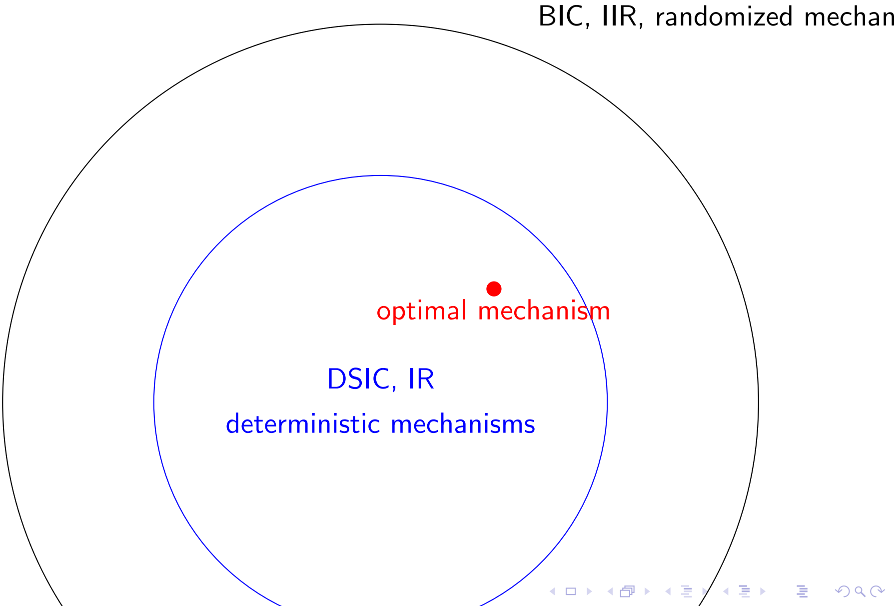
$$a_i^*(v) = 1 \text{ if } w_i(v_i) \geq 0, \; w_i(v_i) \geq w_j(v_j) \; \forall \; j \in N.$$

There exists payments $(p_1, \dots, p_n)$ *such that* $\{a_i, p_i\}_{i \in N}$ *is an optimal mechanism.*

Intepretation

- Every agent has a reserve price - if your value exceeds the reserve price, virtual value becomes non-negative and you are considered for the auction.
- Optimal mechanism is deterministic and DSIC.
- Payments are *cut-off* payments - infimum needed to report so that virtual valuation becomes non-negative and exceeds the second-highest virtual value.

Deterministic and DSIC Optimality



Symmetric Agents

- All β_i are same and all G_i are same.
- Then, virtual valuation functions of all agents become the same - denote it as w.
- For non-negative virtual valuation, agents need to have value greater than or equal to $w^{-1}(0)$.
- Highest value agent is also highest virtual valuation agent.
- So, optimal mechanism is Vickrey auction with a reserve price $w^{-1}(0)$.

An Example

Consider a setting with two agentss whose values are distributed uniformly in the intervals $V_1 = [0, 12]$ (agent 1) and $V_2 = [0, 18]$ (agent 2). Virtual valuation functions of agent 1 and agent 2 are given as:

$$w_1(v_1) = v_1 - \frac{1 - G_1(v_1)}{g_1(v_1)} = v_1 - (12 - v_1) = 2v_1 - 12$$
$$w_2(v_2) = v_2 - \frac{1 - G_2(v_2)}{g_2(v_2)} = v_2 - (18 - v_2) = 2v_2 - 18.$$

Hence, the reserve prices for both the agents are respectively $r_1 = 6$ and $r_2 = 9$.

An Example Continued

The optimal mechanism outcomes are shown for some instances in Table 1.

(v_1, v_2)	Allocation	$p_1(v_1, v_2)$	$p_2(v_1, v_2)$
$(v_1 = 4, v_2 = 8)$	Object not sold	0	0
$(v_1 = 2, v_2 = 12)$	Agent 2	0	9
$(v_1 = 6, v_2 = 6)$	Agent 1	6	0
$(v_1 = 9, v_2 = 9)$	Agent 1	6	0
$(v_1 = 8, v_2 = 15)$	Agent 2	0	11

Table: Description of Optimal Mechanism

Inefficiency

Due to not selling the object.

Due to asymmetric agents - lower valued agent may have higher virtual value if the distributions are not symmetric.

Budget-Balance

A central requirement in many problems is that the payments should add up to zero:

- Bilateral trading: a buyer and a seller exchanging a good.
- Resource sharing: agents collectively sharing a unit of resource.
- Dissolving a partnership: shareholders of a firm redistributing their shares.

General Results

Dominant strategy incentive compatibility, efficiency, and budget-balance are usually incompatible - Green and Laffont.

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The dAGV mechanism is not usually interim individually rational.

In specific settings, it is possible to design Bayesian incentive compatible, efficient, budget-balanced, and interim individually rational mechanisms - dissolving a partnership (Cramton, Gibbons, Klemperer).

Bilateral Trade

- A seller owns a good with value v_s (type) and a buyer wants to buy it with value v_b (type).
- Efficiency: trade when $v_s \leq v_b$ and no trade otherwise.
- Interim IR: Expected utility of buyer is at least 0 and expected utility of seller is at least v_s.
- BB: What buyer pays, seller receives.

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Theorem (Myerson-Satterthwaite)

In the bilateral trading problem, there is no Bayesian incentive compatible, efficient, budget-balanced, and interim individually rational mechanism.

Closing Thoughts

- Convex analysis allows us to solve the single object optimal auction design problem.
- In multidimensional problem, significant progress in understanding the structure of the problem - monotonicity characterization, revenue equivalence extends.
- Optimization in multidimensional problems remain elusive - new approximation techniques from Computer Science (see papers by Chawla and Hartline).
- Budget-balance is difficult to achieve with individual rationality.